Stochastic processes on non-Archimedean spaces. II. Stochastic antiderivational equations.

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Abstract

Stochastic antiderivational equations on Banach spaces over local non-Archimedean fields are investigated. Theorems about existence and uniqueness of the solutions are proved under definite conditions. In particular Wiener processes are considered in relation with the non-Archimedean analog of the Gaussian measure.

1 Introduction.

This article continues investigations of stochastic processes on non-Archimedean spaces ([16]). In the first part stochastic processes were defined on Banach spaces over non-Archimedean local fields and the analogs of Itô formula were proved. This part is devoted to stochastic antiderivational equations. In the non-Archimedean case antiderivational equations are used instead of stochastic integral or differential equations in the classical case.

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Stochastic differential equations on real Banach spaces and manifolds are widely used for solutions of mathematical and physical problems and for construction and investigation of measures on them [5, 11, 13, 26, 27, 28]. Wide classes of quasi-invariant measures including analogous to Gaussian type on non-Archimedean Banach spaces, loops and diffeomorphisms groups were investigated in [17, 18, 20, 23, 24]. Quasi-invariant measures on topological groups and their configuration spaces can be used for the investigations of their unitary representations (see [21, 22, 23, 24] and references therein). In view of this developments non-Archimedean analogs of stochastic equations and diffusion processes need to be investigated. Some steps in this direction were made in [3, 9]. There are different variants for such activity, for example, p-adic parameters analogous to time, but spaces of complex-valued functions. At the same time measures may be real, complex or with values in a non-Archimedean field. In the classical stochastic analysis indefinite integrals are widely used, but in the non-Archimedean case they have quite another meaning, because the field of p-adic numbers $\mathbf{Q_p}$ has not any linear order structure compatible with its normed field structure (see Part I).

This work treats the case which was not considered by another authors and that is suitable and helpful for the investigation of stochastic processes and quasi-invariant measures on non-Archimedean topological groups. In §2 suitable analogs of Gaussian measures are considered. Certainly they have not any complete analogy with the classical one, some of their properties are similar and some are different. They are used for the definiton of the standard (Wiener) stochastic process. Integration by parts formula for the non-Archimedean stochastic processes is studied. Some particular cases of the general Itô formula from Part I are dicussed here more concretely. In §3 with the help of them stochastic antiderivational equations are defined and investigated. Analogs of theorems about existence and uniqiness of solutions of stochastic antiderivational equations are proved. Generating operators of solutions of stochastic equations are investigated. All results of this paper are obtained for the first time.

In this part the notations of Part I also are used.

2 Gaussian measures and standard Wiener processes on a non-Archimedean Banach space.

2.1. Let $H = c_0(\alpha, \mathbf{K})$ be a Banach space over a local field $\mathbf{K}$ with an ordinal α and the standard orthonormal base $\{e_j : j \in \alpha\}$, $e_j = (0, ..., 0, 1, 0, ...)$ with 1 on the j-th place. Let U^P be a cylindrical algebra generated by projections on finite-dimensional over $\mathbf{K}$ subspaces F in H and Borel σ-algebras $Bf(F)$. Denote by U the minimal σ-algebra $\sigma(\mathsf{U}^\mathsf{P})$ generated by U^P. When $card(\alpha) \le \aleph_0$, then $\mathsf{U} = Bf(H)$. Each vector $x \in c_0$ is considered as continuous linear functional on c_0 by the formula $x(y) = \sum_j x^j y^j$ for each $y \in c_0$, so there is the natural embedding $c_0 \hookrightarrow c_0^*$, where $x = \sum_j x^j e_j$, $x^j \in \mathbf{K}$. The field $\mathbf{K}$ is the finite algebraic extension of of the field $\mathbf{Q_p}$ of p-adic numbers and as the Banach space over $\mathbf{Q_p}$ it is isomorphic with $\mathbf{Q_p^n}$, that is, each $z \in \mathbf{K}$ has the form $z = (z^1, ..., z^n)$, where $z^1, ..., z^n \in \mathbf{Q_p}$. Let $\{y\}_p := \sum_{j<0} y_j p^j$, where $y \in \mathbf{Q_p}$, $y = \sum_j y_j p^j$, $y_j \in \{0, 1, ..., p-1\}$, in particular for values $y = (z, x) := \sum_{j=1}^n x^j z^j$ for $x, z \in \mathbf{K}$. All continuous characters $\chi : \mathbf{K} \to \mathbf{C}$ of $\mathbf{K}$ as the additive group have the form

$$(i)\ \chi_\gamma(x) = \epsilon^{z^{-1}\{(e,\gamma x)\}_p}$$

for each $\{(e, \gamma x)\}_p \ne 0$, $\chi_\gamma(x) := 1$ for $\{(e, \gamma x)\}_p = 0$, where $\epsilon = 1^z$ is a root of unity, $z = p^{ord(\{(e,\gamma x)\}_p)}$, $e = (1, ..., 1) \in \mathbf{Q_p^n}$, $\gamma \in \mathbf{K}$ (see §25 [12] and §I.3.6, about the spaces $L_q(H)$ of operators see §I.2). Each χ is locally constant, hence $\chi : \mathbf{K} \to \mathbf{T}$ is also continuous, where $\mathbf{T}$ denotes the discrete group of all roots of 1 (by multiplication).

Let us consider functions, whose Fourier transform has the form:

$$\hat{f}(x) = \hat{f}_{\beta,\gamma,q}(x) := exp(-\beta|x|^q)\chi_\gamma(x),$$

where the Fourier transform was defined in §7 [31] and [29], $\gamma \in \mathbf{K}$, $0 < \beta < \infty$, $0 < q < \infty$.

Definition. A cylindrical measure μ on U^P is called q-Gaussian, if each its one-dimensional projection is q-Gaussian, that is,

$$(i) \quad \mu^g(dx) = C_{\beta,\gamma,q} f_{\beta,\gamma,q} v(dx),$$

where v is the Haar measure on $Bf(\mathbf{K})$ with values in $\mathbf{R}$, where g is a continuous $\mathbf{K}$-linear functional on $H = c_0(\alpha, \mathbf{K})$ giving projection on one-dimensional subspace in H, $C_{\beta,\gamma,q} > 0$ are constants such that $\mu^g(\mathbf{K}) = 1$, β and γ may depend on g, q is independent of g, $1 \le q < \infty$, $\alpha \subset \omega_0$, ω_0 is the first countable ordinal.

If μ is a measure on H, then $\hat{\mu}$ denotes its characteristic functional, that is, $\hat{\mu}(g) := \int_H \chi_g(x)\mu(dx)$, where $g \in H^*$, $\chi_g : H \to \mathbf{C}$ is the character of H as the additive group (see §I.3.6).

2.2. Theorem. *A non-negative q-Gaussian measure μ on $c_0(\omega_0, \mathbf{K})$ is σ-additive on $Bf(c_0(\omega_0, \mathbf{K}))$ if and only if there exists an injective compact operator $J \in L_q(c_0(\omega_0, \mathbf{K}))$ for a chosen $1 \le q < \infty$ such that*

$$(i)\ \mu(dx) = \bigotimes_{j=1}^{\infty} \mu_j(dx^j),\ \textit{where}$$

$$(ii)\ J = diag(\zeta_j :\ \zeta_j \in \mathbf{K},\ j \in \omega_0),$$

$$(iii)\ \mu_j(dx^j) = C_{\beta_j,\gamma_j,q} f_{\beta_j,\gamma_j,q} v(dx^j)$$

are measures on $e_j\mathbf{K}$, $x = (x^j : j \in \omega_0) \in c_0(\omega_0, \mathbf{K})$, $x^j \in \mathbf{K}$, $\beta_j = |\zeta_j|^{-q}$, $\gamma = (\gamma_j : j \in \omega_0) \in c_0(\omega_0, \mathbf{K})$. Moreover, each one-dimensional projection μ^g has the following characteristic functional:

$$(iv) \quad \hat{\mu}^g(h) = exp(-(\sum_j \beta_j |g_j|^q)|h|^q)\chi_{g(\gamma)}(h),$$

where $g = (g_j : j \in \omega_0) \in c_0(\omega_0, \mathbf{K})^*$.

Proof. Let θ be a characteristic functional of μ. By the non-Archimedean analog of the Minlos-Sazonov Theorem (see §2.31 in [18], [19]) a measure μ is σ-additive if and only if for each $c > 0$ there exists a compact operator S_c such that $|Re(\mu(y) - \mu(x))| < c$ for each $x, y \in c_0(\omega_0, \mathbf{K})$ with $|z^* S_c z| < 1$, where $z = x - y$. From the definition of μ to be q-Gaussian it follows, that each its projection μ_j on $\mathbf{K}e_j$ has the form given by Equation (iii). It remains to establish that μ is σ-additive if and only if $J \in L_q(c_0(\omega_0, \mathbf{K}))$ and $\gamma \in c_0(\omega_0, \mathbf{K})$.

In view of Lemma 2.3 [18] μ is σ-additive if and only if each sequence of finite-dimensional (over $\mathbf{K}$ distributions) satisfies two conditions:
$(2.3.i)$ for each $c > 0$ there exists $b > 0$ such that $\sup_n |\ |\mu_{L(n)}|(B(c_0, 0, r) \cap L(n)) - |\mu_{L(n)}|(L(n))| \le c$ for each $r \ge b$,

(2.3.ii) $\sup_n |\mu_{L(n)}|(L(n)) < \infty$. Take in particular $L(n) = sp_{\mathbf{K}}\{e_1, ..., e_n\}$ for each $n \in \mathbf{N}$.

We have $\mu_j(\mathbf{K} \setminus B(\mathbf{K}, 0, r)) \le C \int_{x \in \mathbf{K}, |x| > r} exp(-|x/\zeta_j|^q)|\zeta_j|^{-1} v(dx) \le C_1 \int_{y \in \mathbf{R}, |y| > r} exp(-|y|^q |\zeta_j|^{-q})|\zeta_j|^{-1} dy$, where $C > 0$ and $C_1 > 0$ are constants independent from ζ_j for $b_0 > p^3$ and each $r > b_0$, $1 \le q < \infty$ is fixed (see also the proof of Lemma 2.8 [18] and Theorem II.2.1 [5]). Evidently, $g(\gamma)$ is correctly defined for each $g \in c_0(\omega_0, \mathbf{K})^*$ if and only if $\gamma \in c_0(\omega_0, \mathbf{K})$. In this case the character $\chi_{g(\gamma)} : \mathbf{K} \to \mathbf{C}$ is defined and $\chi_{g(\gamma)} = \prod_{j=1}^{\infty} \chi_{g_j \gamma_j}$. Therefore, if $J \in L_q(c_0)$ and $\gamma \in c_0(\omega_0, \mathbf{K})$, then μ is σ-additive.

Let $0 \ne g \in c_0^*$. Since $\mathbf{K}$ is the local field there exists $x_0 \in c_0$ such that $|g(x_0)| = \|g\|$ and $\|x_0\| = 1$. Put $g_j := g(e_j)$. Then $\|g\| \le \sup_j |g_j|$, since $g(x) = \sum_j x^j g_j$, where $x = x^j e_j := \sum_j x^j e_j$ with $x^j \in \mathbf{K}$. Consequently, $\|g\| = \sup_j |g_j|$. We denumerate the standard orthonormal basis $\{e_j : j \in \mathbf{N}\}$ such that $|g_1| = \|g\|$. There exists an operator E on c_0 with matrix elements $E_{i,j} = \delta_{i,j}$ for each $i, j > 1$, $E_{1,j} = g_j$ for each $j \in \mathbf{N}$. Then $|det\ P_n E P_n| = \|g\|$ for each $n \in \mathbf{N}$, where P_n are the standard projectors on $sp_{\mathbf{K}}\{e_1, ..., e_n\}$. When $g \in \{e_j^* : j \in \omega_0\}$, then evidently, μ^g has the form given by Equation (iii), since $\mu_i(\mathbf{K}) = 1$ for each $i \in \omega_0$, where $e_j^*(e_i) = \delta_{i,j}$ for each i, j.

Suppose now that $J \notin L_q(c_0)$. For this we consider $\mu^g(\mathbf{K} \setminus B(\mathbf{K}, 0, r)) \ge \sum_j \int_{x \in \mathbf{K}, |x| > r} C\ exp(-|x/\zeta_j|^q)|\zeta_j|^{-1} v(dx)$, where $g = (1, 1, 1, ...) \in c_0^* = l^\infty(\omega_0, \mathbf{K})$. On the other hand, there exists a constant $C_2 > 0$ such that for $b_0 > p^3$ and each $r > b_0$ there is the following inequality: $\int_{x \in \mathbf{K}, |x| > r} C\ exp(-|x/\zeta_j|^q)|\zeta_j|^{-1} v(dx) \ge C_2[\int_r^\infty exp(-|y|^q |\zeta_j|^{-q})|\zeta_j|^{-1} dy + \int_{-\infty}^{-r} exp(-|y|^q |\zeta_j|^{-q})|\zeta_j|^{-1} dy]$. From the estimates of Lemma II.1.1 [5] and using the substitution $z = y^{1/2q}$ for $y > 0$ and $z = (-y)^{1/2q}$ for $y < 0$ we get that μ^g is not σ-additive, consequently, μ is not σ-additive, since $P_g^{-1}(A)$ are cylindrical Borel subsets for each $A \in Bf(\mathbf{K})$, where $P_g z = g(z)$ is the induced projection on $\mathbf{K}$ for each $z \in c_0$.

For the verification of Formula (iv) it is sufficient at first to consider the measure μ on the algebra U^{P} of cylindrical subsets in c_0. Then for each projection μ^g, where $g \in sp_{\mathbf{K}}(e_1, ..., e_m)^*$, we have:

$$\hat{\mu}^g(h) = \int_{\mathbf{K}} ... \int_{\mathbf{K}} \chi_e(hz) \mu_1(dx_1) ... \mu_m(dx_m),$$

where $e = (1, ..., 1) \in \mathbf{Q_p^n}$, $h \in \mathbf{K}$, $n := dim_{\mathbf{Q_p}} \mathbf{K}$, $x^i \in \mathbf{K} e_i$, $z = g(x)$, $x = (x^1, ..., x^m)$, consequently, $\hat{\mu}^g(h) = \prod_{i=1}^m \hat{\mu}_i(hg_i)$, since $\chi_e(hg(x)) = \prod_{i=1}^m \chi_e(h_i g_i x^i)$ for each $x \in sp_{\mathbf{K}}(e_1, ..., e_m)$. Since $J \in L_q$, then μ is the Radon measure, consequently, the continuation of μ from U^{P} produces μ on the Borel

σ-algebra of c_0, hence $\lim_{m\to\infty}\hat{\mu}^{Q_m g}(h)=\hat{\mu}^g(h)$, where Q_m is the natural projection on $sp_{\mathbf{K}}(e_1,...,e_m)^*$ for each $m\in\mathbf{N}$ such that $Q_m(g)=(g_1,...,g_m)$. Using expressions of $\hat{\mu}_i$ we get Formula (iv). From this follows, that if $J\in L_q$, then $\hat{\mu}(g)$ exists for each $g\in c_0^*$ if and only if $\gamma\in c_0$, since $\hat{\mu}^g(h)=\hat{\mu}(gh)$ for each $h\in\mathbf{K}$ and $g\in c_0^*$.

2.3. Corollary. *$|\hat{\mu}^g(h_1+h_2)|\le\max(|\hat{\mu}^g(h_1)|,|\hat{\mu}^g(h_2)|)$ for each $h_1,h_2\in\mathbf{K}$ and $g\in c_0(\omega_0,\mathbf{K})^*$.*

Proof. In view of the ultrametric inequality $|h_1+h_2|^q\le\max(|h_1|^q,|h_2|^q)$ for each $1\le q<\infty$ and $h_1,h_2\in\mathbf{K}$. Since $|\chi_\gamma(h)|=1$ for each $h,\gamma\in\mathbf{K}$, then from Formula 2.2.(iv) the statement of this Corollary follows.

2.4. Remark. Let Z be a compact subset without isolated points in a local field $\mathbf{K}$, for example, $Z=B(\mathbf{K},t_0,1)$. Then the Banach space $C^0(Z,\mathbf{K})$ has the Amice polynomial orthonormal base $Q_m(x)$, where $x\in Z$, $m\in\mathbf{N_o}:=\{0,1,2,...\}$ [2]. Suppose $\tilde{P}^{n-1}:C^{n-1}(Z,\mathbf{K})\to C^n(Z,\mathbf{K})$ are antiderivations from §80 [30], where $n\in\mathbf{N}$. Each $f\in C^0$ has a decomposition $f(x)=\sum_m a_m(f)Q_m(x)$, where $a_m\in\mathbf{K}$. These decompositions establish the isometric isomorphism $\theta:C^0(Z,\mathbf{K})\to c_0(\omega_0,\mathbf{K})$ such that $\|f\|_{C^0}=\max_m|a_m(f)|=\|\theta(f)\|_{c_0}$. Since Z is homeomorphic with $\mathbf{Z_p}$, then $\tilde{P}^1\tilde{P}^0:C^0(Z,\mathbf{K})\to C^2(Z,\mathbf{K})$ is a linear injective compact operator such that $\tilde{P}^1\tilde{P}^0\in L_1$, where $\tilde{P}^j$ here corresponds to $\tilde{P}_{j+1}:C^j\to C^{j+1}$ antiderivation operator by Schikhof (see also §§54, 80 [30] and §I.2.1). The Banach space $C^2(Z,\mathbf{K})$ is dense in $C^0(Z,\mathbf{K})$. Using Theorem 2.2 and Note I.2.3 for $q\ge 1$ we get a q-Gaussian measure on $C^0(Z,\mathbf{K})$, where $\tilde{P}^1\tilde{P}^0 f=\sum_j\lambda_jP_jf$ and $Jf=\sum_j\zeta_jP_jf$ for each $f\in C^0$, we put $|\lambda_j||\pi|^q\le|\zeta_j|^q\le|\lambda_j|$ for each $j\in\mathbf{N}$, P_j are projectors, $\lambda_j,\zeta_j\in\mathbf{K}$, $p^{-1}\le|\pi|<1$, $\pi\in\mathbf{K}$ and $|\pi|$ is the generator of the valuation group of $\mathbf{K}$.

If $H=c_0(\omega_0,\mathbf{K})$, then the Banach space $C^0(Z,H)$ is isomorphic with the tensor product $C^0(Z,\mathbf{K})\otimes H$ (see §4.R [29]). Therefore, the antiderivation $\tilde{P}^n$ on $C^n(Z,\mathbf{K})$ induces the antiderivation $\tilde{P}^n$ on $C^n(Z,H)$. If $J_i\in L_q(Y_i)$, then $J:=J_1\otimes J_2\in L_q(Y_1\otimes Y_2)$ (see also Theorem 4.33 [29]). Put $Y_1=C^0(Z,\mathbf{K})$ and $Y_2=H$, then each $J:=J_1\otimes J_2\in L_q(Y_1\otimes Y_2)$ induces the q-Gaussian measure μ on $C^0(Z,H)$ such that $\mu=\mu_1\otimes\mu_2$, where μ_i are q-Gaussian measures on Y_i induced by J_i as above. In particular for $q=1$ we also can take $J_1=\tilde{P}^1\tilde{P}^0$. The 1-Gaussian measure on $C^0(Z,H)$ induced by $J=J_1\otimes J_2\in L_1$ with $J_1=\tilde{P}^1\tilde{P}^0$ we call standard. Analogously considering the following Banach subspace $C^0_0(Z,H):=\{f\in C^0(Z,H):f(t_0)=0\}$ and operators $J:=J_1\otimes J_2\in L_1(C^0_0(Z,\mathbf{K})\otimes H)$ we get the 1-Gaussian measures

μ on it also, where $t_0 \in Z$ is a marked point. Certainly, we can take others operators $J_1 \in L_q(Y_1)$ not related with the antiderivation as above.

3 Non-Archimedean stochastic antiderivational equations.

3.1. A measurable space (Ω, F) with a normalised non-negative measure λ is called a probability space and is denoted by $(\Omega, \mathsf{F}, \lambda)$, where F is a σ-algebra of Ω. Points $\omega \in \Omega$ are called elementary events and values $\lambda(S)$ are called probabilities of events $S \in \mathsf{F}$. A measurable map $\xi : (\Omega, \mathsf{F}) \to (X, \mathsf{B})$ is called a random variable with values in X, where B is a σ-algebra of X (see §I.4.1).

3.2. We define a (non-Archimedean) Wiener process $w(t, \omega)$ with values in H as a stochastic process such that:

(i) the differences $w(t_4, \omega) - w(t_3, \omega)$ and $w(t_2, \omega) - w(t_1, \omega)$ are independent for each chosen ω, (t_1, t_2) and (t_3, t_4) with $t_1 \ne t_2$, $t_3 \ne t_4$, either t_1 or t_2 is not in the two-element set $\{t_3, t_4\}$, where $\omega \in \Omega$;

(ii) the random variable $\omega(t, \omega) - \omega(u, \omega)$ has a distribution $\mu^{F_{t,u}}$, where μ is a probability Gaussian measure on $C^0(T, H)$ described in §§2.1, 2.4, $\mu^g(A) := \mu(g^{-1}(A))$ for $g \in C^0(T, H)^*$ and each $A \in Bf(C^0(T, H))$, a continuous linear functional $F_{t,u}$ is given by the formula $F_{t,u}(w) := w(t, \omega) - w(u, \omega)$ for each $w \in L^s(\Omega, \mathsf{F}, \lambda; C^0_0(T, H))$, where $1 \le s \le \infty$;

(iii) we also put $w(0, \omega) = 0$, that is, we consider a Banach subspace $L^s(\Omega, \mathsf{F}, \lambda; C^0_0(T, H))$ of $L^s(\Omega, \mathsf{F}, \lambda; C^0(T, H))$, where $\Omega \ne \emptyset$.

If μ is not a Gaussian measure on $C^0_0(T, H)$ and a stochastic process w satisfies conditions $(i - iii)$, then it is called the (non-Archimedean) stochastic process (see §I.4.2). If μ is the standard Gaussian measure on $C^0_0(T, H)$, then the Wiener process is called standard (see also Theorem 3.23, Lemmas 2.3, 2.5, 2.8 and §3.30 in [18]).

3.3. Remark. In Part I the non-Archimedean analogs of the Itô formula were proved. In the particular case $H = \mathbf{K}$ we have $a \in L^s(\Omega, \mathsf{F}, \lambda; C^0(T, \mathbf{K}))$, $E \in L^r(\Omega, \mathsf{F}, \lambda; C^0(T, \mathbf{K}))$, $f \in C^n(T \times \mathbf{K}, Y)$ and $w \in L^q(\Omega, \mathsf{F}, \lambda; C^0_0(T, \mathbf{K}))$ are functions (see §§4.2, 4.6 [16] and §3.2), so that

$$\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[(\partial^{m+b} f / \partial u^b \partial x^m)(u, \xi(u, \omega)) \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})]|_{u=t} =$$

$$\sum_j (\partial^{m+b} f / \partial u^b \partial x^m)(t_j, \xi(t_j, \omega))[t_{j+1} - t_j]^{b+m-l} a(t_j, \omega)^{k-l} [E(t_j, \omega)(w(t_{j+1}, \omega) - w(t_j, \omega))]^l$$

for each $m + b \le n$, where $t_j = \sigma_j(t)$, $a(t, \omega)$, $E(t, \omega)$ and $w(t, \omega) \in \mathbf{K}$, that is a, E, w commute. In particular $\tilde{P}^m_{u,0} f(u) = \sum_{k=1}^m (k!)^{-1} \hat{P}_{u^k} f^{(k)}(u)$, that is $\tilde{P}^m_{u,0} f(u)|_{u=t} = \tilde{P}_{m+1} f'(t)$, where $\tilde{P}_{m+1} : C^m(T, \mathbf{K}) \to C^{m+1}(T, \mathbf{K})$ is the Schikhof linear continuous antiderivation operator (see for comparison §80 [30]).

In the classical case measures are real-valued and functions ϕ are with values in Banach spaces over $\mathbf{R}$ or $\mathbf{C}$. But in the considered here case measures are real-valued and functions are with values in Banach spaces over non-Archimedean fields $\mathbf{K}$, so the mean value $M\|f\|$ is real and not with values in $\mathbf{K}$. This leads to differences with the classical case, in particular formula $M[(\int_S^T \phi(t, \omega) dB_t(\omega))^2] = M[\int_S^T \phi(t, \omega)^2 dt]$ (see Lemma 3.5 [28]) is not valid, but there exists its another analog. Let X be a locally compact Hausdorff space and $BC_c(X, H)$ denotes a subspace of $C^0(X, H)$ consisting of bounded continuous functions f such that for each $\epsilon > 0$ there exists a compact subset $V \subset X$ for which $\|f(u)\|_H < \epsilon$ for each $u \in X \setminus V$. In particular for $X \subset \mathbf{K}$, $e^* \in H^*$ and a fixed $t \in X$ in accordance with Theorem 7.22 [29] there exists a $\mathbf{K}$-valued tight measure $\mu_{t,\omega,e^*,b,k}$ on the σ-algebra $Bco(X)$ of clopen subsets in X such that $e^* \hat{P}_{u^b, w^k} \psi(u, x, \omega) \circ (I^{\otimes b} \otimes E^{\otimes k})|_{u=t} = \int_X \psi(u, E(u, \omega) w(u, \omega), \omega) \mu_{t,\omega,e^*,b,k}(du)$ for each $\psi \in L^r(\Omega, \mathsf{F}, \lambda; BC_c(X, L_k(H^{\otimes k}, H)))$ and $E \in L^q(\Omega, \mathsf{F}, \lambda; BC_c(X, L(H)))$, where H^* is a topologically conjugate space, $1 \le r, q \le \infty$, $1/r + 1/q \ge 1$.

If $\chi_\gamma : \mathbf{K} \to S^1 := \{z \in \mathbf{C} : |z| = 1\}$ is a continuous character of $\mathbf{K}$ as the additive group, then $M\chi_\gamma((e^* \hat{P}_{u^b, w^k} \psi(u, x, \omega) \circ (I^{\otimes b} \otimes E^{\otimes k})|_{u=t})^l) = \prod_j M\chi_\gamma((e^* \psi(t_j, x, \omega)[t_{j+1} - t_j]^b \circ (1^{\otimes b} \otimes (E(t_j, \omega)[w(t_{j+1}, \omega) - w(t_j, \omega)])^{\otimes k})^l)$ due to Condition $I.4.2.(i)$. For ψ independent from x, $l = 1$, $k = 2$, $b = 0$, $E = 1$ and $H = \mathbf{K}$ (so that $e^* = 1$) it takes a simpler form, which can be considered as another analog of the classical formula. For the evaluation of appearing integrals tables from §1.5.5 [31] can be used. Another important result is the following theorem.

Theorem. *Let $\psi \in L^2(\Omega, \mathsf{F}, \lambda; C^0(T, L(H)))$, $w \in L^2(\Omega, \mathsf{F}, \lambda; C^0_0(T, H))$ be the stochastic process on the Banach space H over $\mathbf{K}$. Then there exists a function $\phi \in C^0(T, H)$ such that $M\chi_\gamma(g\hat{P}_{w(u,\omega)} \psi(u, \omega) \circ I|_{u=t}) = \hat{\mu}(\gamma g \hat{P}_u \phi(u)|_{u=t})$ for each $\gamma \in \mathbf{K}$ and each $t \in T$ and for each $g \in H^*$.*

Proof. Let $t \in T$ and $t_j = \sigma_j(t)$, where σ_j is the approximation of the identity in T, $F_{a,b}(w) := w(a, \omega) - w(b, \omega)$ for $a, b \in T$ (see §I.2.1 [16] and §3.2). In view of Conditions $I.4.2.(i, ii)$ and the Hahn-Banach theorem

(see [29]) there exists a projection operator Pr_g such that $\hat{\mu}^{(F_{a,b}gE)}(h) = \hat{\mu}^{(F_{a,b}Pr_g)}(Pr_gEh)$, since $F_{a,b}ghEw = ghE(w(a,\omega)-w(b,\omega)) = hgEF_{a,b}w$ for each $a,b \in T$ and for each $h \in \mathbf{K}$, where $\hat{\mu}$ is the characteristic functional of the measure μ corresponding to w, that is, $\hat{\mu}(g) := \int_{C^0_0(T,H)} \chi_g(y)\mu(dy)$, where $g \in C^0_0(T,H)^*$, $\chi_g : C^0_0(T,H) \to \mathbf{C}$ is the character of $C^0_0(T,H)$ as the additive group, $E \in L(H)$, $y \in C^0_0(T,H)$, μ is the Borel measure on $C^0_0(T,H)$ (see also §I.3.6). The random variable $E(w(a,\omega)-w(b,\omega))$ has the distribution $\mu^{F_{a,b}E}$ for each $a \ne b \in T$ and $E \in L(H)$. On the other hand the projection operator Pr_e commutes with the antiderivation operator $\hat{P}_u$ on $C^0(T,H)$, where $(Pr_ef)(t) := Pr_ef(t)$ is defined pointwise for each $f \in C^0(T,H)$. In $L^2(\Omega,\mathsf{F},\lambda;C^0(T,H))$ the family of step functions $f(t,\omega) = \sum_{j=1}^n Ch_{U_j}(\omega)f_j(t)$ is dense, where $f_j \in C^0(T,H)$, Ch_U is the characteristic function of $U \in \mathsf{F}$, $n \in \mathbf{N}$, since $\lambda(\Omega)=1$ and λ is nonnegative. For each $t \in T$ there exists $\lim_{j\to\infty} \psi(t_j,\omega).(w(t_{j+1},\omega)-w(t_j,\omega))$ in $L^2(\Omega,\mathsf{F},\lambda;H)$ (see Theorem I.2.14).

If $A \in L(H)$, then

(i) $\chi_\gamma((g_1+g_2)Az) = \chi_\gamma(g_1Az)\chi_\gamma(g_2Az)$ for each $g_1,g_2 \in H^*$ and $z \in H$,

(ii) $\chi_\gamma(gA(z_1+z_2)) = \chi_\gamma(gAz_1)\chi_\gamma(gAz_2)$ for each $g \in H^*$ and $z_1,z_2 \in H$,

(iii) $\chi_\gamma(agAz) = [\chi_\gamma(gAz)]^{\zeta(a)}$ for each $\{(e,\gamma gAz)\}_p \ne 0$ and $a \in \mathbf{K}$, where $\zeta(a) := \{(e,\gamma agAz)\}_p/\{(e,\gamma gAz)\}_p$. On the other hand A is completely defined by the family $\{e^*_iAe_j : i,j \in \alpha\}$, where $H = c_0(\alpha,\mathbf{K})$, $e^*_i(e_j) = \delta_{i,j}$, $e^*_i \in H^*$, $\{e_j : j \in \alpha\}$ is the standard orthonormal base of H. Hence the family $\{\chi_\gamma(ae^*_iAe_j) : i,j \in \alpha; a \in \mathbf{K}\}$ completely characterize $A \in L(H)$ due to Equations $(i-iii)$, when $\gamma \ne 0$.

For each $y \in H$ and each $\gamma \in \mathbf{K}$ the function $M\chi_\gamma(g\psi(t,\omega)y)$ is continuous by $t \in T$, consequently, there exists a continuous function $\phi : T \to H$ such that $M\chi_\gamma(g\psi(t,\omega)y) = \chi_\gamma(g\phi(t)y)$ for each $y \in H$ and $t \in T$, since characters χ_γ are continuous from $\mathbf{K}$ to $\mathbf{C}$ and $\chi_\gamma(h) = \chi_1(\gamma h)$ for each $0 \ne \gamma \in \mathbf{K}$ and $h \in \mathbf{K}$ and the $\mathbf{C}$-linear span of the family $\{\chi_\gamma : \gamma \in \mathbf{K}\}$ of characters is dense in $C^0(\mathbf{K},\mathbf{C})$ by the Stone-Weierstrass theorem [10]. On the other hand, $\lim_{j\to\infty} \chi_\gamma(\sum_{i=0}^j a_j) = \prod_{i=1}^\infty \chi_\gamma(a_i)$, when $\lim_j a_j = 0$ for a sequence a_j in $\mathbf{K}$. Therefore,

$$M\chi_\gamma(g\sum_{j=0}^\infty \psi(t_j,\omega).[w(t_{j+1},\omega)-w(t_j,\omega)]) = \prod_{j=0}^\infty \hat{\mu}(\gamma g\phi(t_j)(t_{j+1}-t_j))$$

$$= \hat{\mu}(\gamma g\hat{P}_u\phi(u)|_{u=t}) \quad \text{for each } t \in T \text{ and each } g \in H^*.$$

From the equality $\chi_{a+b}(c)=\chi_a(c)\chi_b(c)$ for each a,b and $c\in \mathbf{K}$ the statement of this theorem follows for each $\gamma \in \mathbf{K}$.

3.4. Theorem. *Let* $a\in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))))$ *and* $E\in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)))))$, $a=a(t,\omega,\xi)$, $E=E(t,\omega,\xi)$, $t\in B_R$, $\omega\in\Omega$, $\xi\in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))$ *and* $\xi_0\in L^q(\Omega,\mathsf{F},\lambda;H)$, *and* $w\in L^q(\Omega,\mathsf{F},\lambda;C^0_0(B_R,H))$, *where a and E satisfy the local Lipschitz condition:*

(LLC) *for each* $0<r<\infty$ *there exists* $K_r>0$ *such that* $\max(\|a(t,\omega,x)-a(t,\omega,y)\|,\|E(t,\omega,x)-E(t,\omega,y)\|)\le K_r\|x-y\|$ *for each* $x,y\in B(C^0(B_R,H),0,r)$ *and* $t\in B_R$, $\omega\in\Omega$, $1\le q\le\infty$. *Then the stochastic process of the following type:*

(i) $\xi(t,\omega)=\xi_0(\omega)+(\hat{P}_u a)(u,\omega,\xi)|_{u=t}+(\hat{P}_{w(u,\omega)}E)(u,\omega,\xi)|_{u=t}$ *has the unique solution.*

Proof. We have $\max(\|a(x)-a(y)\|^g,\|E(x)-E(y)\|^g)\le K\|x-y\|^g$, hence $\max(\|a(x)\|^g,\|E(x)\|^g)\le K_1(\|x\|^g+1)$ for each $x,y\in H$ and for each $1\le g<\infty$ and each $t\in B_R$ and each $\omega\in\Omega$, where K and K_1 are positive constants, $a(x)$ and $E(x)$ are short notations of $a(t,\omega,x)$ and $E(t,\omega,x)$ for $x=\xi(t,\omega)$ respectively. For solving equation (i) we use iterations:

$X_0(t)=x$,..., $X_n(t)=x+\hat{P}_u a(X_{n-1}(u))|_{u=t}+\hat{P}_w E(X_{n-1}(u))|_{u=t}$, consequently, $X_{n+1}-X_n(t)=I_1(t)+I_2(t)$, where $I_1(t)=\hat{P}_u[a(X_n(u))-a(X_{n-1}(u))]|_{u=t}$, $I_2(t)=\hat{P}_w[E(X_n(u))-E(X_{n-1}(u))]|_{u=t}$, $x(t)$ and $X_n(t)$ are short notations of $x(t,\omega)$ and $X_n(t,\omega)$ respectively. Let $M\eta$ be a mean value of a real-valued distribution $\eta(\omega)$ by $\omega\in\Omega$, where $(\Omega,\mathsf{F},\lambda)$ is the probabilty space, then $M\|P_u[a(X_n(u))-a(X_{n-1}(u))]|_{u=t}\|^g\le K(M\|P_t\|^g)M\sup_u\|X_n(u)-X_{n-1}(u)\|^g$, where $X_n\in L^q(\Omega,\mathsf{F},\lambda;C^0_0(B_R,H))$ for each n, since $|\lambda|(\Omega)=1$ and $\|x\|_\infty=\sup_{1\le g<\infty}\|x\|_g=ess-\sup_{\omega\in\Omega}\|x(\omega)\|_H$ for $x\in L^\infty(\Omega,\mathsf{F},\lambda;H)$. While $1\le q<\infty$ we put $g=q$, for $q=\infty$ we take $ess-\sup$. Also

$$M\|\hat{P}_w[E(X_n(u))-E(X_{n-1}(u))]|_{u=t}\|^g\le K\|\hat{P}_w\|^gM\sup_u\|X_n(u)-X_{n-1}(u)\|^g$$

$$\le (K\|\hat{P}_w\|^g)^lM\sup_u\|X_{n-l+1}(u)-X_{n-l}(u)\|^g$$

in particular for $l=n-1$. On the other hand,

$$X_1(t)=x(t)+\hat{P}_u a(x(u))|_{u=t}+\hat{P}_w E(x(u))|_{u=t},$$

consequently, $\|X_1(t)-X_0(t)\|^g\le\max(\|[P_u a(x(u))]|_{u=t}\|^g,\|\hat{P}_w E(x(u))|_{u=t_0}\|^g)$, where $w(0)=0$, $\hat{P}_u a(u)|_{u=t_0}=0$, $\hat{P}_w E|_{u=t_0}=0$. For each $\epsilon>0$ there exists

$B_\epsilon \subset B_R$ such that $K\|\hat{P}_w|_{B_\epsilon}\|^g < 1$ and $K\|P_t|_{B_\epsilon}\|^g < 1$. Therefore, there exists the unique solution on each B_ϵ, since $\sup_u \|X_1(u) - X_0(u)\|^g < \infty$ and $\lim_{l\to\infty}(K\|\hat{P}_w|_{B_\epsilon}\|^g)^l C = 0$, $\lim_{l\to\infty}(K\|P_t|_{B_\epsilon}\|^g)^l C = 0$, hence there exists $\lim_{n\to\infty} X_n(t) = X(t) = \xi(t,\omega)|_{B_\epsilon}$, where $C := M\sup_{u\in B_\epsilon}\|X_1(u) - X_0(u)\|^g \le \max(\|\hat{P}_w\|^g, \|P_t\|^g)(\|x\|^g_{C^0}+1)K < \infty$, here B_ϵ is an arbitrary ball of radius ϵ in B_R, $t \in B_\epsilon$.

If X^1 and X^2 are two solutions, then $X^1 - X^2 =: \psi = \sum_{j=1}^n C_j Ch_{B(\mathbf{K},x_j,r_j)}$, where $n \in \mathbf{N}$, $C_j \in \mathbf{K}$, $T = B_R$, since B_R has a disjoint covering by balls $B(\mathbf{K},x_j,r_j)$, on each such ball there exists the unique solution with a given initial condition on it (that is, in a chosen point x_j such that C_j and $B(\mathbf{K},x_j,r_j)$ are independent from ω). Therefore, $\psi = \hat{P}_u[a(u,X^2) - a(u,X^1)]|_{u=t} + \hat{P}_w[E(u,X^2) - E(u,X^1)]|_{u=t}$, hence $\Phi^1\psi(t_i;1;t_{i+1}-t_i) = [a(t_i,X^2(t_i)) - a(t_i,X^1(t_i)) + [E(t_i,X^2(t_i)) - E(t_i,X^1(t_i))][w(t_{i+1}) - w(t_i)]/(t_{i+1} - t_i)$ for each $t_i \ne t_{i+1}$, $t_i = \sigma_i(t)$ due to Condition $I.2.1.(ii)$, where $w(t)$ is the short notation of $w(t,\omega)$. The term $(\Phi^1 w)(t_i;1;t_{i+1}-t_i) = [w(t_{i+1}) - w(t_i)]/(t_{i+1}-t_i)$ has the infinite-dimensional over $\mathbf{K}$ range in $C^0(B_R^2 \setminus \Delta, H)$ for each $\omega \in \Omega$, where $\Delta := \{(u,u) : u \in B_R\}$. If $(\Phi^1 w) = 0$, then $a(t,X^2(t,\omega)) - a(t,X^1(t,\omega)) = 0$. If $a(t,X^2) = a(t,X^1)$ for each t and almost all ω, then $\hat{P}_w[E(t,X^2(t)) - E(t,X^1(t))] = 0$ which is possible only for $\psi = 0$. If $a(t,X^2) \ne a(t,X^1)$ and the function ψ is locally constant by t and independent from ω, then $\hat{P}_u[a(u,X^1+g) - a(u,X^1)]|_{u=t} + \hat{P}_w[E(u,X^1+g) - E(u,X^1)]|_{u=t}$ is locally constant by t and independent from ω only for $[a(u,X^2) - a(u,X^1)] = 0$ and $[E(u,X^2) - E(u,X^1)] = 0$ due to definitions of $\hat{P}_u$ and $\hat{P}_w$, hence $\psi = 0$, since it is evident for $a(u,X)$ and $E(u,X)$ depending on X locally polynomially or polyhomogeneously for each u, but such locally polynomial or polyhomogeneous functions by X are dense in

$$L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)))) \text{ and}$$

$$L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))))) \text{ respectively.}$$

3.5. Theorem. *Let* $a \in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))))$ *and* $E \in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)))))$, $a = a(t,\omega,\xi)$, $E = E(t,\omega,\xi)$, $t \in B_R$, $\omega \in \Omega$, $\xi \in L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))$ *and* $\xi_0 \in L^q(\Omega,\mathsf{F},\lambda;H)$, *where a and E satisfy the local Lipschitz condition (see 3.4.(LLC)). A stochastic process of the type*

(i) $\xi(t,\omega) = \xi_0(\omega) + \sum_{m+b=1}^\infty \sum_{l=0}^m (\hat{P}_{u^{b+m-l},w(u,\omega)^l}[a_{m-l+b,l}(u,\xi(u,\omega))\circ(I^{\otimes b}\otimes a^{\otimes(m-l)}\otimes E^{\otimes l})])|_{u=t}$

such that $a_{m-l,l} \in C^0(B_{R_1} \times B(L^q(\Omega, \mathsf{F}, \lambda; C^0(B_R, H)), 0, R_2), L_m(H^{\otimes m}; H))$ *(continuous and bounded on its domain) for each* n, l, $0 < R_2 < \infty$ *and*

(ii) $\lim_{n\to\infty} \sup_{0\le l\le n} \|a_{n-l,l}\|_{C^0(B_{R_1}\times B(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)),0,R_2),L_n(H^{\otimes n},H))} = 0$ *for each* $0 < R_1 \le R$ *when* $0 < R < \infty$, *or each* $0 < R_1 < R$ *when* $R = \infty$, *for each* $0 < R_2 < \infty$.

Then (i) *has the unique solution in* B_R.

Proof. Let $X_0(t) = x$,...,

$$X_n(t) = x + \sum_{m+b=1}^{\infty} \sum_{l=0}^{m} (\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[a_{m-l+b,l}(u, X_{n-1}(u,\omega)) \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t},$$

consequently,

$$X_{n+1} - X_n(t) = \sum_{m+b=1}^{\infty} \sum_{l=0}^{m} (\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[a_{m-l+b,l}(u, X_n(u)) - a_{m-l+b,l}(u, X_{n-1}(u))]$$

$\circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t}$,
where in general $\hat{P}_{a(u,\xi)}1|_{u=t} = a(t, \xi(t,\omega)) - a(t_0, \xi(t_0,\omega)) \ne \hat{P}_u a(u,\xi) = \sum_j a(t_j, \xi(t_j,\omega))[t_{j+1} - t_j]$, $t_j = \sigma_j(t)$ for each $j = 0, 1, 2, \ldots$. Then

$$M\|\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[a_{m-l+b,l}(u, X_n(u)) - a_{m-l+b,l}(u, X_{n-1}(u))]|_{(B_{R_1}\times B(L^q,0,R_2))} \circ ($$

$$I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t}\|^g \le K(M\|\hat{P}_{u^{b+m-l}, w(u,\omega)^l}\|^g)\|a_{m-l+b,l}|_{(B_{R_1}\times B(L^q,0,R_2))}\|^g$$

$$(M \sup_u \|X_n(u) - X_{n-1}(u)\|^g)(M \sup_u \|a\|^{m-l})(M \sup_u \|E\|^l),$$

where $X_n \in C^0_0(B_R, H)$ for each n, K is the same constant as in §3.4, $1 \le g < \infty$. On the other hand,

$$X_1(t) = x(t) + \sum_{m+b=1}^{\infty} \sum_{l=0}^{m} (\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[a_{m-l+b,l}(u, x(u)) \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t},$$

consequently,

$$\|X_1(t) - X_0(t)\|^g \le \sup_{m,l,b} (\|\hat{P}_{u^{b+m-l}, w(u,\omega)^l}[a_{m-l+b,l}(u, x(u)) \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t}\|^g.$$

Due to Condition (ii) for each $\epsilon > 0$ and $0 < R_2 < \infty$ there exists $B_\epsilon \subset B_R$ such that

$$K \sup_{m,l,b} (\|\hat{P}_{u^{b+m-l}, w(u,\omega)^l}|_{B_\epsilon}[a_{m-l+b,l}(u, *)|_{(B_\epsilon \times B(L^q,0,R_2))} \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])\|^g =: c < 1.$$

Therefore, there exists the unique solution on each B_ϵ, since $\sup_u \|X_1(u) - X_0(u)\| < \infty$ and $\lim_{l\to\infty} c^l C = 0$ for each $C > 0$, hence there exists $\lim_{n\to\infty} X_n(t) = X(t) = \xi(t,\omega)|_{B_\epsilon}$, where $C := M \sup_{u\in B_\epsilon} \|X_1(u) - X_0(u)\|^g \le (c+1)K < \infty$, here B_ϵ is an arbitrary ball of radius ϵ in B_R, $t \in B_\epsilon$.

If X^1 and X^2 are two solutions, then $X^1 - X^2 =: \psi = \sum_{j=1}^n C_j Ch_{B(\mathbf{K},x_j,r_j)}$ as in §3.4. If S is a polyhomogeneous function, then there exists $n = deg(S) < \infty$ such that differentials $D^m S = 0$ for each $m > n$, but its antiderivative $\hat{P}$ has $D^{n+1}\hat{P}S \ne 0$. If $\|S_1\| > \|S_2\|$, then $\|\hat{P}S_1\| > \|\hat{P}S_2\|$, which we can apply to a convergent series considering terms $\|D^m\hat{P}S\|(mod\ p^k)$ for each $k \in \mathbf{N}$. Therefore,

$\psi = \sum_{m+b=1}^\infty \sum_{l=0}^m (\hat{P}_{u^{b+m-l},w(u,\omega)^l}[a_{m-l+b,l}(u,X^2) - a_{m-l+b,l}(u,X^1)] \circ (I^{\otimes b} \otimes a^{\otimes(m-l)} \otimes E^{\otimes l})])|_{u=t}$, where the function ψ is locally constant by t and independent from ω, hence $\psi = 0$, since it is evident for $a(u,X)$ and $E(u,X)$ and $a_{k-l,l}(u,X)$ depending on X locally polynomially or polyhomogeneously for each u, but such locally polynomial or polyhomogeneous functions by X are dense in

$$L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)))) \text{ and}$$

$$L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,L(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H))))) \text{ and}$$

$$C^0(B_{R_1} \times B(L^q(\Omega,\mathsf{F},\lambda;C^0(B_R,H)),0,R_2), L_k(H^{\otimes k};H))$$

respectively.

3.6. Proposition. *Let ξ be the Wiener process given by Equation 3.4.(i) with the 1-Gaussian measure associated with the operator $\tilde{P}^1\tilde{P}^0$ as in §2.4 and let also $\max(\|a(t,\omega,x) - a(v,\omega,x)\|, \|E(t,\omega,x) - E(v,\omega,x)\|) \le |t-v|(C_1 + C_2\|x\|^b)$ for each t and $v \in B(\mathbf{K},t_0,R)$ λ-almost everywhere by $\omega \in \Omega$, where b, C_1 and C_2 are non-negative constants. Then ξ with probability 1 has a C^2-modification and $q(t) \le \max\{\|\xi_0\|^s, |t-t_0|(C_1 + C_2 q(t))\}$ for each $t \in B(\mathbf{K},t_0,R)$, where $q(t) := \sup_{|u-t_0|\le|t-t_0|} M\|\xi(t,\omega)\|^s$ and $\mathbf{N} \ni s \ge b \ge 0$.*

Proof. For the following function $f(t,x) = x^s$ in accordance with Theorem I.4.6 [16] we have $f(t,\xi(t,\omega)) = f(t_0,\xi_0)+$

$$+\sum_{k=1}^s \sum_{l=0}^k \binom{k}{l} (\hat{P}_{u^{k-l},w(u,\omega)^l}[(\binom{s}{k}\xi(t,\omega)^{s-k}(u,\xi(u,\omega)) \circ (a^{\otimes(k-l)} \otimes E^{\otimes l})])|_{u=t},$$

hence $M\|\xi(t,\omega)\|^s \le \max(\|\xi_0\|^s, |t-t_0| d(\hat{P}^s_*)(C_1 + C_2 \sup_{|u-t_0|\le|t-t_0|} M\|\xi(u,\omega)\|^s)$, since $|t_j - t_0| \le |t-t_0|$ for each $j \in \mathbf{N}$ and $M\|\xi(t,\omega) - \xi(v,\omega)\|^s \le |t-v|(1+$

$C_1+C_2d(\hat{P}^s_*)\sup_{|u-t_0|\le\max(|t-t_0|,|v-t_0|)} M\|\xi(u,\omega)\|^s)$, since $|t_j-v_j|\le|t-v|+\rho^j$ for each $j\in\mathbf{N}$, where $0<\rho<1$,

$$d(\hat{P}^s_*) := \sup_{a\ne0,E\ne0,f\ne0}\max_{s\ge k\ge l\ge0}\|(k!)^{-1}\binom{k}{l}\hat{P}_{u^{k-l},w^l}(\partial^k f/\partial^k x)\circ(a^{\otimes(k-l)}\otimes E^{\otimes l})\|/$$

$$(\|a\|^{k-l}_{C^0(B_R,H)}\|E\|^l_{C^0(B_R,L(H))}\|f\|_{C^s(B_R,H)}),$$

hence $d(\hat{P}^s_*)\le 1$, since $f\in C^s$ as a function by x and $(\bar{\Phi}^s g)(x;h_1,...,h_s;0,...,0)=D^s_x g(x).(h_1,...,h_s)/s!$ for each $g\in C^s$ and due to the definition of $\|g\|_{C^s}$. Considering in particular polyhomogeneous g on which $d(\hat{P}^s_*)$ takes its maximum value we get $d(\hat{P}^s_*)=1$. Since $P(C^2)=1$ for the Markov measure P induced by the transition measures $P(v,x,t,S):=\mu^{F_{t,v}}(S|\xi(v)=x)$ for $t\ne v$ of the non-Archimedean Wiener process (see §2.2), then ξ has with the probability 1 a C^2-modification.

Note. If to consider a general stochastic process as in §I.4.3, then from the proof of Proposition 3.6 it follows, that ξ with the probability 1 has a modification in the space $J(C^0_0(T,H))$, where J is a nondegenerate correlation operator of the product measure μ on $C^0_0(T,H)$.

3.7. Proposition. *Let ξ be a stochastic process given by Equation 3.4.(i) and $\max(\|a(t,\omega,x_1)-a(v,\omega,x_2)\|,\|E(t,\omega,x_1)-E(v,\omega,x_2)\|)\le|t-v|(C_1+C_2\|x_1-x_2\|^b)$ for each t and $v\in B(\mathbf{K},t_0,R)$ λ-almost everywhere by $\omega\in\Omega$, where b, C_1 and C_2 are non-negative constants. Then two solutions ξ_1 and ξ_2 with initial conditions $\xi_{1,0}$ and $\xi_{2,0}$ satisfy the following inequality: $y(t)\le\max\{\|\xi_{1,0}-\xi_{2,0}\|^s,|t-t_0|(C_1+C_2y(t))\}$ for each $t\in B(\mathbf{K},t_0,R)$, where $y(t):=\sup_{|u-t_0|\le|t-t_0|}M\|\xi_1(t,\omega)-\xi_2(t,\omega)\|^s$ and $\mathbf{N}\ni s\ge b\ge0$.*

Proof. From §3.6 it follows, that $M\|\xi_1(t,\omega)-\xi_2(t,\omega)\|^s\le|t-t_0|(C_1+C_2\sup_{|u-t_0|\le|t-t_0|}M\|\xi_1(u,\omega)-\xi_2(u,\omega)\|^s)$, since $d(\hat{P}^s_*)\le1$.

3.8. Remark. Let $X_t=X_0+\hat{P}_ta+\hat{P}_wv$ and $Y_t=Y_0+\hat{P}_tq+\hat{P}_ws$ be two stochastic processes corresponding to $E=I$ and a Banach algebra H over $\mathbf{K}$ in §I.4.6 [16, 4, 29]. Then $X_uY_u-X_tY_t=(X_u-X_t)(Y_u-Y_t)+X_t(Y_u-Y_t)+(X_u-X_t)Y_t$, where $u,t\in T$. Hence $d(X_tY_t)=X_tdY_t+(dX_t)Y_t+(dX_t)(dY_t)$. Therefore,

$$\hat{P}_{X_t}Y_t=X_tY_t-X_0Y_0-\hat{P}_{Y_t}X_t-\hat{P}_{(X_t,Y_t)}1,$$

which is the non-Archimedean analog of the integration by parts formula, where in all terms X_t is displayed on the left from Y_t. For two C^1 functions f and g we have $(fg)'=f'g+fg'$ or $d(fg)=gdf+fdg$, that is terms

with $(dt)(dt)$ are absent, consequently, $(dt)(dt) = 0$. In a particular case $X_t = Y_t = w_t$ this leads two $w_t^2 - w_0^2 - 2\hat{P}_{w_t} w_t = \hat{P}_{(w_t,w_t)}1$, where the last term corresponds two $(dw_t)(dw_t) \ne 0$. This means that

$$d(w^2) = 2wdw + (dw)(dw).$$

For $X_t = w_t$ and $Y_t = t$ the integration by parts formula gives $\hat{P}_{w_t} t = w_t t - \hat{P}_t w_t - \hat{P}_{(t,w_t)}1$. Such that $\hat{P}_{(t,w_t)}1 = \sum_j t_j [w_{t_{j+1}} - w_{t_j}] - w_t t + \sum_j w_{t_j} [t_{j+1} - t_j] \ne 0$, for example, for $t = 1$, $w \in C^0_0(T, H)$, $T = \mathbf{Z_p}$ and $t_0 = 0$ this gives $\hat{P}_{(t,w_t)}1 = w_1 - w_0 = w_1$. Therefore, $(dt)(dw_t) \ne 0$, that is the important difference of the non-Archimedean and classical cases (see for comparison Exer. 4.3 and Theorem 4.5 [28]).

If H is a Banach space over the local field $\mathbf{K}$ and $f(x, y) = x^*y$ is a $\mathbf{K}$-bilinear functional on it, where x^* is an image of $x \in H$ under an embedding $H \hookrightarrow H^*$ associated with the standard orthonormal base $\{e_j\}$ in H, then

$$\hat{P}_{X_t^*} Y_t = X_t^* Y_t - X_0^* Y_0 - \hat{P}_{Y_t^*} X_t - \hat{P}_{(X_t^*, Y_t)}1,$$

hence $d(X_t^* Y_t) = X_t^* dY_t + (dX_t^*) Y_t + (dX_t^*)(dY_t)$ and $d(w^*w) = w^*dw + (dw^*)w + (dw^*)(dw)$.

3.9. Definition. If $\xi(t, \omega) \in L^q(\Omega, \mathsf{F}, \lambda; C^0(B_R, H)) =: Z$ is a stochastic process and $T(t, s)$ is a family of bounded linear operators satisfying the following Conditions $(i - iv)$:

(i) $T(t, s) : H_s \to H_t$, where $H_s := L^q(\Omega, \mathsf{F}, \lambda; C^0(B(\mathbf{K}, 0, |s|), H))$,

(ii) $T(t, t) = I$,

(iii) $T(t, s)T(s, v) = T(t, v)$ for each $t, s, v \in B_R$,

(iv) $M_s\{\|T(t, s)\eta\|^q_H\} \le C\|\eta\|^q_H$ for each $\eta \in H_s$, where C is a positive nonrandom constant, $1 \le q \le \infty$, then $T(t, s)$ is called a multiplicative operator functional of the stochastic process ξ.

If $T(t, s; \omega)$ is a system of random variables on Ω with values in $L(H)$, satisfying almost surely Conditions $(i - iii)$ and uniformly by $t, s \in B_R$ Condition (iv) such that

(v) $(T(t, s)\eta)(\omega) = T(t, s; \omega)\eta(\omega)$, then such multiplicative operator functional is called homogeneous. An operator

(vi) $A(t) = \lim_{s\to 0}[T(t, t + s) - I]/s$ is called the generating operator of the evolution family $T(t, v)$. If $T(t, v) = T(t, v; \omega)$ depends on ω, then $A(t) = A(t; \omega)$ is also considered as the random variable on Ω (depending on the parameter ω) with values in $L(H)$.

3.10. Remark. Let $A(t)$ be a linear continuous operator on a Banach space Y over $\mathbf{K}$ such that it depends strongly continuously on $t \in B(\mathbf{K}, 0, R)$, that is $A(t)y$ is continuous by t for each chosen $y \in Y$ and $A(t) \in L(Y)$. Then the solution of the differential equation

(1) $dx(t)/dt = A(t)x(t)$, $x(s) = x_0$ has a solution

(2) $x(t) = U(t,s)x(s)$, where $U(t,s)$ is a generating operator such that

(3) $U(t,s) = I + \hat{P}_u A(u)U(u,s)|_{u=s}^{u=t}$, though $x(t)$ may be non-unique, where $x(s) = x_0$ is an initial condition, $x, t \in B(\mathbf{K}, 0, R)$. The solution of Equation (3) exists using the method of iterations (see §3.4). Indeed, in view of Lemma I.2.3 [16] $U(s,s) = I$ and

(4) $dx(t)/dt = \partial U(t,s)x(s)/\partial t = A(t)U(t,s)x(s) = A(t)x(t)$. If to consider a solution of the antiderivational equation

(5) $V(t,s) = I + \hat{P}_u V(t,u)A(u)|_{u=s}^{u=t}$, then it is a solution of the Cauchy problem

(6) $\partial V(t,s)/\partial s = -V(t,s)A(s)$, $V(t,t) = I$. Therefore, $\partial[V(t,s)U(s,v)]/\partial s = -V(t,s)A(s)U(s,v) + V(t,s)A(s)U(s,v) = 0$, hence $V(t,s)U(s,v)$ is not dependent from s, consequently, there exist U and V such that

(7) $V(t,s) = U(t,s)$ for each $t, s \in B(\mathbf{K}, 0, R)$. From this it follows, that

(8) $U(t,s)U(s,u) = U(t,u)$ for each $s, u, t \in B(\mathbf{K}, 0, R)$. In particular, if $A(t) = A$ is a constant operator, then there exists a solution $U(t,s) = EXP((t-s)A)$ (see about EXP in Proposition 45.6 [30]). Equation (3) has a solution under milder conditions, for example, $A(t)$ is weakly continuous, that is $e^*A(t)\eta$ is continuous for each $e^* \in Y^*$ and $\eta \in Y$, then $e^*U(t,s)\eta$ is differentiable by t and $U(t,s)$ satisfies Equation (4) in the weak sense and there exists a weak solution of (5) coinciding with $U(t,s)$. If to substitute $A(t)$ on another operator $\tilde{A}(t)$, then for the corresponding evolution operator $\tilde{U}(t,s)$ there is the following inequality:

(9) $\|\tilde{U}(t,s) - U(t,s)\| \le M\tilde{M} \sup_{u \in B(\mathbf{K},0,R)} \|\tilde{A}(u) - A(u)\| R$, where $M := 1 + \sup_{s,t \in B(\mathbf{K},0,R)} \|U(t,s)\|$ and $\tilde{M}$ is for $\tilde{U}$.

Proposition. *Let $B(t)$ and two sequences $A_n(t)$ and $B_n(t)$ be given of strongly continuous on $B(\mathbf{K}, 0, R)$ bounded linear operators and $\tilde{U}(t,s)$ be evolution operators corresponding to $\tilde{A}_n(t) = A_n(t) + B_n(t)$, where $\sup_{n \in \mathbf{N}, u \in B(\mathbf{K},0,R)} \|B_n(u)\| \le \sup_{u \in B(\mathbf{K},0,R)} \|B(u)\| = C < \infty$. If $MCR < 1$, then there exists a sequence $\tilde{U}_n(t,s)$ which is also uniformly bounded. If there exists $U_n(t,s)$ strongly and uniformly converging to $U(t,s)$ in $B(\mathbf{K}, 0, R)$, then $\tilde{U}_n(t,s)$ also can be chosen strongly and uniformly convergent.*

Proof. From the use of Equations $(3, 8)$ iteratively for $U_n(\sigma_{j+1}(t), \sigma_j(t))$ and $U_n(\sigma_j(t), s)$ and also for $\tilde{U}_n$ and taking $\tilde{U}_n - U_n$ it follows, that

(10) $\tilde{U}_n(t, s) = U_n(t, s) + \hat{P}_v U_n(t, v) B_n(v) \tilde{U}_n(v, s)|_{v=s}^{v=t}$ for each $n \in \mathbf{N}$. Therefore, $\|\tilde{U}_n(t, s)\| \le M + MC \sup_v \|\tilde{U}_n(v, s)\| R$, hence $\|\tilde{U}_n(t, s)\| \le M/[1 - MCR]$, since $MCR < 1$. If $\lim_n x_n = x$ in Y and $U_n(t, s)x$ is uniformly convergent to $U(t, s)x$, then for each $\epsilon > 0$ there exist $\delta > 0$ and $m \in \mathbf{N}$ such that $\sup_{t,s \in B(\mathbf{K},0,R)} \|U_n(t+h, s+v)x_n - U_n(t, s)x_n\| < \epsilon$ for each $n > m$ and $\max(|h|, |v|) < \delta$ due to Equality (10).

3.11. Proposition. *Let a, $a_{m-l+b,l}$ and E be the same as in §3.5. Then Equation 3.5.(i) has the unique solution ξ in B_R for each initial value $\xi(t_0, \omega) \in L^q(\Omega, \mathsf{F}, \lambda; H)$ and it can be represented in the following form:*

(2) $\xi(t, \omega) = T(t, t_0; \omega)\xi(t_0; \omega)$, *where $T(t, v; \omega)$ is the multiplicative operator functional.*

Proof. In view of Theorem 3.5, Definition 3.9, Remark and Proposition 3.10 with the use of a parameter $\omega \in \Omega$ the statement of Proposition 3.11 follows.

3.12. Let now consider the case $J(C_0^0(T, H)) \subset C^1(T, H)$ (see §3.6), for example, the standard Wiener process.

Corollary. *Let a function $f(t, x)$ satisfies conditions of §I.4.8 [16], then a generating operator of an evolution family $T(t, v)$ of a stochastic process $\eta = f(t, \xi(t, \omega))$ is given by the following equation:*

(1) $A(t)\eta(t) = f'_t(t, \xi(t, \omega)) + f'_x(t, \xi(t, \omega)) \circ a(t, \omega) +$

$$f'_x(t, \xi(t, \omega)) \circ E(t, \omega) {w'}_t(t, \omega) + \sum_{m+b \ge 2, 0 \le m \in \mathbf{Z}, 0 \le b \in \mathbf{Z}} ((m+b)!)^{-1} \sum_{l=0}^{m} \binom{m+b}{m} \binom{m}{l}$$

$$\{(b+m-l)(\hat{P}_{u^{b+m-l-1}, w(u,\omega)^l}[(\partial^{(m+b)} f/\partial u^b \partial x^m)(u, \xi(u, \omega)) \circ (I^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes l})])|_{u=t} +$$
$$l(\hat{P}_{u^{b+m-l}, w(u,\omega)^{l-1}}[(\partial^{(m+b)} f/\partial u^b \partial x^m)(u, \xi(u, \omega)) \circ (I^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes (l-1)})] E {w'}_u(u, \omega))|_{u=t}\}.$$

Proof. In view of Theorem I.4.8 [16] and Proposition 3.11 there exists a generating operator of an evolution family. From Lemma I.2.3 and Formula I.4.8.(ii) [16] it follows the statement of this Corollary.

Remark. If $f(t, x)$ satisfies conditions either of §I.4.6 or of §I.4.7, then Formula 3.12.(1) takes simpler forms, since the corresponding terms vanish.

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